Review of Rational Functions

A rational function is a function that can be written as the ratio of two polynomials where the denominator isn't zero.

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0}$$

Domain

The domain of a rational function is all real values except where the denominator, q(x) = 0.

Roots

The roots, zeros, solutions, x-intercepts (whatever you want to call them) of the rational function will be the places where p(x) = 0.
**Set the top = 0 and solve.*

That is, completely ignore the denominator. Whatever makes the numerator zero will be the roots of the rational function, just like they were the roots of the polynomial function earlier.

If you can write it in factored form, then you can tell whether it will cross or touch the x-axis at each x-intercept by whether the multiplicity on the factor is odd or even.
Example: $(x - 2)^2$ is even, will touch to form double root, $(x - 1)^3$ is odd, will cross for single root.

Vertical Asymptotes

An asymptote is a line that the curve approaches but does not cross. The equations of the vertical asymptotes can be found by finding the roots of q(x).
**Set the bottom = 0 and solve*

Completely ignore the numerator when looking for vertical asymptotes, only the denominator matters.

If you can write it in factored form, then you can tell whether the graph will be asymptotic in the same direction or in different directions by whether the multiplicity is even or odd.

Asymptotic in the same direction means that the curve will go up or down on both the left and right sides of the vertical asymptote. Asymptotic in different directions means that the one side of the curve will go down and the other side of the curve will go up at the vertical asymptote.

Horizontal Asymptotes

A horizontal line is an asymptote only to the far left and the far right of the graph. "Far" left or "far" right is defined as anything past the vertical asymptotes or x-intercepts.

Horizontal asymptotes are not asymptotic in the middle. It is okay to cross a horizontal asymptote in the middle.

The location of the horizontal asymptote is determined by looking at the degrees of the numerator (m) and denominator (n).

- *If m < n, then y = 0 is the horizontal asymptote.*
- *If m = n, then y =* $\frac{a_m}{b_n}$ *is the horizontal asymptote. That is, the ratio of the leading coefficients.*
- *If m > n, there is no horizontal asymptote. However, if m = n + 1, there is an oblique or slant asymptote.*

Holes

Sometimes, a factor will appear in the numerator and in the denominator. Let's assume the factor (x-k) is in the numerator and denominator. Because the factor is in the denominator, x=k will not be in the domain of the function. This means that one of two things can happen.
There will either be a vertical asymptote at x=k, or there will be a hole at x=k.

Let's look at what will happen in each of these cases.

- *There are more (x-k) factors in the denominator*.
  After dividing out all duplicate factors, the (x-k) is still in the denominator. Factors in the denominator result in vertical asymptotes. Therefore, there will be a vertical asymptote at x=k.

- *There are more (x-k) factors in the numerator*.
  After dividing out all the duplicate factors, the (x-k) is still in the numerator. Factors in the numerator result in x-intercepts. But, because you can't use x=k, there will be a hole in the graph on the x-axis.

- *There are equal numbers of (x-k) factors in the numerator and denominator*.
  After dividing out all the factors (because there are equal amounts), there is no (x-k) left at all. Because there is no (x-k) in the denominator, there is no vertical asymptote at x=k. Because there is no (x-k) in the numerator, there is no x-intercept at x=k. There is just a hole in the graph, someplace other than on the x-axis. To find the exact location, plug in x=k into the reduced function (you can't plug it into the original, it's undefined, there), and see what y-value you get.

Oblique Asymptotes

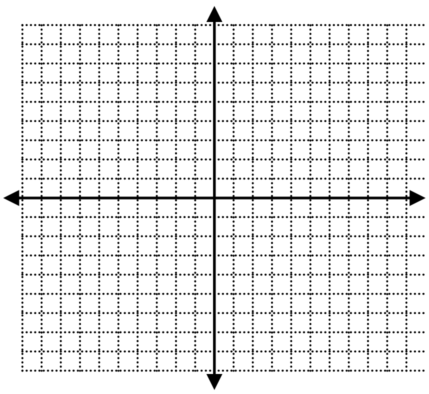
When the degree of the numerator is exactly one more than the degree of the denominator (m = n + 1), the graph of the rational function will have an oblique asymptote. Another name for an oblique asymptote is a slant asymptote.

To find the equation of the oblique asymptote, perform long division (synthetic if it will work) by dividing the denominator into the numerator. As x gets very large (this is the far left or far right that I was talking about), the remainder portion becomes very small, almost zero. So, to find the equation of the oblique asymptote, perform the long division and discard the remainder.

Examples:

Sketch each graph and determine domain & range, find all roots and asymptotes.

1. $f(x) = \dfrac{x^2+1}{x^2+x-2}$



2. $f(x) = \dfrac{x^3-3x^2-4x}{x^2+3x}$.